

Power counting of various Dirac covariants in hadronic Bethe-Salpeter wave functions for decay constant calculations of pseudoscalar mesons

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Abstract

We have employed the framework of Bethe-Salpeter equation under covariant instantaneous ansatz to calculate leptonic decay constants of unequal mass pseudoscalar mesons like π^\pm , K , D , D_s and B and radiative decay constants of neutral pseudoscalar mesons like π^0 and η_c in two photons. In the Dirac structure of hadronic Bethe-Salpeter wave function, the covariants are incorporated from their complete set in accordance with a recently proposed power counting rule. The decay constants are calculated with the incorporation of both Leading order and Next-to-leading order Dirac covariants. The results validate the power counting rule which provides a practical means of incorporating Dirac covariants in the Bethe-Salpeter wave function for a hadron.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is the theory to describe strong interactions. However, the large gauge coupling at low energies (long distances) destroys the perturbative expansion. As a result, many non-perturbative approaches have been proposed to deal with this long distance properties of QCD, such as QCD sum rules, Lattice QCD, dynamical-equation-based approaches like Schwinger-Dyson equation and Bethe-Salpeter equation (BSE), and potential models. Since the task of calculating hadron structures from QCD itself is very difficult, as can be seen from various Lattice QCD approaches, one generally relies on specific models to gain some understanding of QCD at low energies. BSE is a conventional approach in dealing with relativistic bound state problems. From the solutions we can obtain useful information about the inner structure of hadrons, which is also crucial in treating high energy hadronic scatterings. The BSE framework is firmly rooted in field theory, and provides a realistic description for analyzing hadrons as composite objects. Despite its drawback of having to input model-dependent kernel, these studies have become an interesting topic in recent years, since calculations have shown that BSE framework using phenomenological potentials can give satisfactory results on more and more data being accumulated.

In this paper we study leptonic decays of pseudoscalar mesons (P-mesons) such as π , K , D , D_S and B , which proceed through the coupling of quark-antiquark loop to the axial vector current and also the two-photon decays of neutral pseudoscalar mesons such as π^0 and η_c which proceed through the famous quark-triangle diagrams. We employ QCD motivated BSE under Covariant Instantaneous Ansatz (CIA) in this paper [1, 2]. CIA is a Lorentz-invariant generalization of Instantaneous Ansatz. For a $q\bar{q}$ system, the CIA formulation ensures an exact interconnection between 3D and 4D forms of BSE [2, 3]. The 3D form of BSE serves for making contact with the mass spectrum, whereas the 4D form provides the Hadron-quark vertex function for evaluation of various hadronic transition amplitudes through quark loop diagrams. In these studies one of the main ingredients is the Dirac structure of the Bethe-Salpeter wave function (BSW). The copious Dirac structure of BSW was already studied by Llewellyn Smith [4] much earlier. Recent studies [5, 6] have revealed that various covariant structures in BSWs of various hadrons is necessary to obtain quantitatively accurate observables. It has been further noticed that all covariants do not contribute equally for calculation of meson observables. So it is interesting to investigate

how to arrange these covariants. In a recent work [2], we developed a power counting rule for incorporating various Dirac structures in BSW, order-by-order in powers of inverse of meson mass. We have outlined the Dirac covariants and expanded the coefficients to the leading order (LO), and calculated the leptonic decay constants of vector mesons (ρ , ω , ϕ , ψ) [2] as well as pseudoscalar mesons (π , K , D , D_S and B) [3] at this order. The results agree with data well.

However, common to all the perturbative theories, it is better to calculate the next order(s) to the leading one and make sure it is (they are) really smaller w.r.t. the LO, before claiming the validation of the perturbation. At the same time, as more and preciser data accumulated, it is useful to arrange more available parameters inherent in our framework to accommodate better fits to gain more precise information of the structure of hadron. So the study of next-to-leading order (NLO) is natural and essential. For all the mesons, the pseudoscalar is the simplest in Dirac structure. As the first step, we collect the data of leptonic decay constants f_P 's for pseudoscalar mesons (π , K , D , D_S and B), to fit three parameters B'_i 's in our framework at NLO. We found: a) NLO works better than LO. b) NLO corrections are smaller than those of LO (π is exceptional for its small mass, to be discussed later in this paper). Then with the fitted parameters we calculate the radiative decay constants F_P of neutral pseudoscalar mesons, π^0 and η_c at NLO. We also found satisfying agreement with data, and fair improvement w.r.t. LO. Thus the fact that three parameters can give a good fit not only for 5 different cases of f_P , but also giving satisfactory results for two cases of F_P , demonstrates the validity and robustness of this framework. These results indicate that our power counting scheme [2] provides a practical means of incorporating various Dirac structures from their complete set into the BS wave function.

In what follows, we give a detailed discussion of the fit and calculation at NLO, after a brief review of our framework. The paper is organized as follows: In section II, we discuss the structure of BS wave function for P-mesons in BSE under CIA using the power counting rule. In section III, we introduce the fitting to f_P for pseudoscalar mesons. The radiative decay constants F_P for π^0 and η_c mesons are calculated in section 4, while we conclude with Discussion in section V.

II. THE BSW UNDER CIA

A. BSE under CIA

We first outline the BSE framework under CIA. We have employed for the case of scalar quarks for simplicity. For a $q\bar{q}$ system with an effective kernel K and 4D wave function $\Phi(P, q)$, the 4D BSE takes the form,

$$i(2\pi)^4 \Delta_1 \Delta_2 \Phi(P, q) = \int d^4 q K(q, q') \Phi(P, q'), \quad (1)$$

where $\Delta_{1,2} = m_{1,2}^2 + p_{1,2}^2$ are the inverse propagators of two scalar quarks, and $m_{1,2}$ are (effective) constituent masses of quarks. The 4-momenta of the quark and anti-quark, $p_{1,2}$, are related to the internal 4-momentum q_μ and total momentum P_μ of hadron of mass M as

$$p_{1,2\mu} = \hat{m}_{1,2} P_\mu \pm q_\mu, \quad (2)$$

where $\hat{m}_{1,2} = [1 \pm (m_1^2 - m_2^2)/M^2]/2$ are the Wightman-Garding (WG) definitions of masses of individual quarks. Now it is convenient to express the internal momentum of the hadron q as the sum of two parts, the transverse component, $\hat{q}_\mu = q_\mu - \frac{q \cdot P}{P^2} P_\mu$ which is orthogonal to total hadron momentum P (ie. $\hat{q} \cdot P = 0$ regardless of whether the individual quarks are on-shell or off-shell), and the longitudinal component, $\sigma P_\mu = (q \cdot P / P^2) P_\mu$, which is parallel to P . We now use an Ansatz on the BS kernel K in Eq. (1) which is assumed to depend on the 3D variables $\hat{q}_\mu, \hat{q}'_\mu$ [7] i.e.

$$K(q, q') = K(\hat{q}, \hat{q}'), \quad (3)$$

A similar form of the BS kernel was also earlier suggested in ref. [8]). Hence, the longitudinal component, σP_μ of q_μ , does not appear in the form $K(\hat{q}, \hat{q}')$ of the kernel. For reducing Eq.(1) to the 3D form, we define a 3D wave function $\phi(\hat{q})$ as

$$\phi(\hat{q}) = \int_{-\infty}^{+\infty} M d\sigma \Phi(P, q). \quad (4)$$

Substituting Eq. (4) in Eq. (1), with definition of kernel in Eq. (3), we get a covariant version of Salpeter equation,

$$(2\pi)^3 D(\hat{q}) \phi(\hat{q}) = \int d^3 \hat{q}' K(\hat{q}, \hat{q}') \phi(\hat{q}'), \quad (5)$$

where $D(\hat{q})$ is the 3D denominator function defined by

$$\frac{1}{D(\hat{q})} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1 \Delta_2}, \quad (6)$$

whose value can be easily worked out by contour integration by noting positions of poles in the complex σ -plane (shown in detail in [9]) as,

$$D(\hat{q}) = \frac{(\omega_1 + \omega_2)^2 - M^2}{\frac{1}{2\omega_1} + \frac{1}{2\omega_2}}, \quad \omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2. \quad (7)$$

We can see that RHS of Eq. (5) is identical to RHS of Eq. (1) by virtue of Equations (3) and (4). We thus have an exact interconnection between 3D wave function $\phi(\hat{q})$ and 4D wave function $\Phi(P, q)$:

$$\Delta_1 \Delta_2 \Phi(P, q) = \frac{D(\hat{q}) \phi(\hat{q})}{2\pi i} \equiv \Gamma(\hat{q}). \quad (8)$$

We also get the $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ under CIA for case of scalar quarks. Further in the process, an exact interconnection between 3D and 4D BSE [7] is thus brought out where the 3D form serves for making contact with the mass spectrum of hadrons, whereas the 4D form provides the vertex $Hq\bar{q}$ function $\Gamma(\hat{q})$ which satisfies a 4D BSE with a natural off-shell extension over the entire 4D space (due to the positive definiteness of the quantity $\hat{q}^2 = q^2 - (q \cdot P)^2 / P^2$ throughout the entire 4D space) and thus provides a fully Lorentz-invariant basis for evaluation of various transition amplitudes through various quark loop diagrams.

B. Dirac structure of Hadron-quark vertex function for P-mesons in BSE with power counting scheme

To obtain the form of Hadron-quark vertex function for the case of fermionic quarks constituting a particular meson, we first replace the scalar propagators Δ_i^{-1} in Eq. (7) by the proper fermionic propagators S_F . The $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ now is a 4×4 matrix in spinor space. For incorporation of the relevant Dirac structures in $\Gamma(\hat{q})$, we make use of the power counting rule we developed in [2], order-by-order in powers of inverse of meson mass [2]. Our aim of developing the power counting rule was to find a “criterion” so as to systematically choose among various Dirac covariants from their complete set to write wave functions for different mesons (vector mesons, pseudoscalar mesons etc.).

As far as a pseudoscalar meson is concerned, its hadron-quark vertex function which has a certain dimensionality of mass can be expressed as a linear combination of four Dirac covariants [4], each multiplying a Lorentz scalar amplitude, as function of $q \cdot P$. We note that in the expression for CIA vertex function in equation (7), the factor $D(\hat{q})\phi(\hat{q})$ is nothing but the Lorentz-invariant momentum dependent scalar which depends on q^2 , P^2 and $q \cdot P$ and has a certain dimensionality of mass. However the Lorentz-scalar amplitudes multiplying various Dirac structures in [5] have different dimensionalities of mass. For adapting this decomposition to write the structure of $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ for a particular meson, we re-express this function by making these scalar amplitudes dimensionless by weighing each covariant by an appropriate power M , the meson mass. Thus each term in the expansion of $\Gamma(\hat{q})$ is associated with a certain power of M and hence in detail we can express the hadron-quark vertex, $\Gamma(\hat{q})$ as a polynomial in various powers of $1/M$:

$$\Gamma^P(\hat{q}) = \Omega^P \frac{1}{2\pi i} N_P D(\hat{q}) \phi(\hat{q}), \quad (9)$$

with

$$\Omega^P = \gamma_5 B_0 - i\gamma_5(\gamma \cdot P) \frac{B_1}{M} - i\gamma_5(\gamma \cdot q) \frac{B_2}{M} - \gamma_5[(\gamma \cdot P)(\gamma \cdot q) - (\gamma \cdot q)(\gamma \cdot P)] \frac{B_3}{M^2}, \quad (10)$$

where B_i ($i = 0, \dots, 3$) are four dimensionless coefficients to be determined. Since we use constituent quark masses, where quark mass m is approximately half of the hadron mass

M , we can use the ansatz

$$q \ll P \sim M \quad (11)$$

in the rest frame of the hadron (however we wish to mention that among all the pseudoscalar mesons, pion enjoys the special status in view of its unusually small mass ($M < \Lambda_{QCD}$) and its case should be considered separately). Then each of the four terms in Eq. (9,10) would again receive suppression by different powers of $1/M$. Thus we can arrange these terms as an expansion in powers of $O(1/M)$. We can then see in the expansion of Ω^P , that the structures associated with the coefficients B_0 , B_1 have magnitudes $O(1/M^0)$ and are of leading order, while those with B_2 , B_3 are $O(1/M^1)$ and are next-to-leading-order. This naïve power counting rule suggests that the maximum contribution to the calculation of any pseudoscalar meson observable should come from the Dirac structures γ_5 and $i\gamma_5(\gamma \cdot P)/M$ associated with the constant coefficients B_0 and B_1 respectively, followed by the other two higher order covariants associated with coefficients B_2 and B_3 . In general, the coefficients B_i of the Dirac structures could be functions of $q \cdot P$, and hence can be written as a Taylor series in powers of $q \cdot P$. However the coefficients used here are dimensionless on lines of [2]. So they are in fact function of $q \cdot P/M^2$. Then the leading order contribution of the coefficients are the case when the B_i 's are constant. In this paper, we assume the coefficients are smooth functions of $q \cdot P/M^2$, so to NLO, we only consider the terms of eq.(10), with the coefficients B_i constant. Because the normalization of the BSW can be fixed (see below), B_0 here can be taken to be 1. So we totally have 3 parameters to be fitted at NLO, comparing to one parameter at LO. In a similar manner one can express the full hadron-quark vertex function for a scalar and axial vector meson also in BSE under CIA. At the same time, the restriction by charge parity on wave function of eigenstate should also be respected. Further, to get the complete set of the Dirac structures for a certain kind of meson, the restriction by the (space) Parity have been employed; and it is easy to see that the requirements of the space Parity and the charge Parity are the same for the vertex as well as the full wavefunction [10]. In this work to calculate the leptonic and radiative decay constants, we take the form of hadron-quark vertex as in Eqs. (9) and (10) which incorporates LO as well as NLO covariants and see the relative importance of various covariants.

C. BSE Kernel and the scalar wave function

From the above analysis of the structure of vertex function $Hq\bar{q}$, we notice that the structure of 3D wave function $\phi(\hat{q})$ as well as the form of the 3D BSE are left untouched and have the same form as in our previous works which justifies the usage of the same form of the input kernel we used earlier [2]. Now we briefly mention some features of the BS formulation employed. The structure of BSE is characterized by a single effective kernel arising out of a four-fermion lagrangian in the Nambu-Jonalasino [11, 12] sense. The formalism is fully consistent with Nambu-Jona-Lasino [11] picture of chiral symmetry breaking but is additionally Lorentz-invariant because of the unique properties of the quantity \hat{q}^2 , which is positive definite throughout the entire 4D space. The input kernel $K(q, q')$ in BSE is taken as one-gluon-exchange like as regards color $[(\boldsymbol{\lambda}^{(1)}/2) \cdot (\boldsymbol{\lambda}^{(2)}/2)]$ and spin $(\gamma_\mu^{(1)}\gamma_\mu^{(2)})$ dependence. The scalar function $V(q - q')$ is a sum of one-gluon exchange V_{OGE} and a confining term V_{conf} . Thus we can write the interaction kernel as [2, 12]:

$$K(q, q') = \left(\frac{1}{2}\boldsymbol{\lambda}^{(1)}\right) \cdot \left(\frac{1}{2}\boldsymbol{\lambda}^{(2)}\right) V_\mu^{(1)} V_\mu^{(2)} V(q - q');$$

$$V_\mu^{(1,2)} = \pm 2m_{1,2}\gamma_\mu^{(1,2)};$$

$$V(\hat{q} - \hat{q}') = \frac{4\pi\alpha_S(Q^2)}{(\hat{q} - \hat{q}')^2} + \frac{3}{4}\omega_{q\bar{q}}^2 \int d^3\mathbf{r} \left[r^2(1 + 4a_0\hat{m}_1\hat{m}_2M^2r^2)^{-1/2} - \frac{C_0}{\omega_0^2} \right] e^{i(\hat{q}-\hat{q}')\cdot\mathbf{r}}; \quad (12)$$

$$\alpha_S(Q^2) = \frac{12\pi}{33 - 2f} \left(\ln \frac{M_{>}^2}{\Lambda^2} \right)^{-1}; \quad M_{>} = \text{Max}(M, m_1 + m_2).$$

The Ansatz employed for the spring constant $\omega_{q\bar{q}}^2$ in Eq. (12) is [2, 12],

$$\omega_{q\bar{q}}^2 = 4\hat{m}_1\hat{m}_2M_{>}\omega_0^2\alpha_S(M_{>}^2), \quad (13)$$

where \hat{m}_1, \hat{m}_2 are the Wightman-Garding definitions of masses of constituent quarks defined earlier. Here the proportionality of $\omega_{q\bar{q}}^2$ on $\alpha_S(Q^2)$ is needed to provide a more direct QCD motivation to confinement. This assumption further facilitates a flavour variation in $\omega_{q\bar{q}}^2$. And ω_0^2 in Eq. (12) and Eq. (13) is postulated as a universal spring constant which is

common to all flavours. Here in the expression for $V(\hat{q} - \hat{q}')$, as far as the integrand of the confining term V_{conf} is concerned, the constant term C_0/ω_0^2 is designed to take account of the correct zero point energies, while a_0 term ($a_0 \ll 1$) simulates an effect of an almost linear confinement for heavy quark sectors (large m_1, m_2), while retaining the harmonic form for light quark sectors (small m_1, m_2) [12] as is believed to be true for QCD. Hence the term $r^2(1 + 4a_0\hat{m}_1\hat{m}_2M_{>}^2r^2)^{-1/2}$ in the above expression is responsible for effecting a smooth transition from harmonic ($q\bar{q}$) to linear ($Q\bar{Q}$) confinement. The basic input parameters in the kernel are just four i.e. $a_0 = 0.028$, $C_0 = 0.29$, $\omega_0 = 0.158$ GeV and QCD length scale $\Lambda = 0.20$ GeV and quark masses, $m_{u,d} = 0.265$ GeV, $m_s = 0.415$ GeV, $m_c = 1.530$ GeV and $m_b = 4.900$ GeV which have been earlier fit to the mass spectrum of $q\bar{q}$ mesons[12] obtained by solving the 3D BSE under Null-Plane Ansatz (NPA). However due to the fact that the 3D BSE under CIA has a structure which is formally equivalent to the 3D BSE under NPA, near the surface $P \cdot q = 0$, the $q\bar{q}$ mass spectral results in CIA formalism are exactly the same as the corresponding results under NPA formalism[9, 12]. The details of BS model under CIA in respect of spectroscopy are thus directly taken over from NPA formalism (see [2, 9, 12]. Now comes to the problem of the 3D BS wave function. The ground state wave function $\phi(\hat{q})$ satisfies the 3D BSE on the surface $P \cdot q = 0$, which is appropriate for making contact with O(3)-like mass spectrum (see [12]). Its fuller structure is reducible to that of a 3D harmonic oscillator with coefficients dependent on the hadron mass M and the total quantum number N . The ground state wave function $\phi(\hat{q})$ deducible from this equation thus has a gaussian structure [2, 12] and is expressible as:

$$\phi(\hat{q}) \sim e^{-\hat{q}^2/2\beta^2}. \quad (14)$$

In the structure of $\phi(\hat{q})$ in (14), the parameter β is the inverse range parameter which incorporates the content of BS dynamics and is dependent on the input kernel $K(q, q')$. The structure of the parameter β in $\phi(\hat{q})$ is taken as [2, 9, 12]:

$$\beta^2 = (2\hat{m}_1\hat{m}_2M\omega_{q\bar{q}}^2/\gamma^2)^{1/2}; \gamma^2 = 1 - \frac{2\omega_{q\bar{q}}^2C_0}{M_{>}\omega_0^2}. \quad (15)$$

We now give the calculation of leptonic decays constants of pseudoscalar mesons employing both LO and NLO Dirac covariants according to our power counting scheme in the

framework discussed in next section.

III. CALCULATIONS AND RESULTS FOR f_P

A. Leptonic decays of pseudoscalar mesons to NLO

Decay constants f_P can be evaluated through the loop diagram which gives the coupling of the two-quark loop to the axial vector current and can be evaluated as:

$$f_P P_\mu = \langle 0 | \bar{Q} i \gamma_\mu \gamma_5 Q | P(P) \rangle, \quad (16)$$

which can in turn be expressed as a loop integral,

$$f_P P_\mu = \sqrt{3} \int d^4 q \text{Tr}[\Psi_P(P, q) i \gamma_\mu \gamma_5]. \quad (17)$$

Bethe-Salpeter wave function $\Psi(P, q)$ for a P-meson is expressed as,

$$\Psi(P, q) = S_F(p_1) \Gamma(\hat{q}) S_F(-p_2), \quad (18)$$

which is expressed as the quark and anti-quark propagators flanking the Hadron-quark vertex $\Gamma(\hat{q})$ function which is in turn expressed by Eq. (9,10).

Using $\Psi(P, q)$ from Eq. (18), and incorporating $Hq\bar{q}$ vertex function $\Gamma(\hat{q})$ from Eq. (9,10) in Eq. (17), evaluating trace over the gamma matrices and multiplying both sides of Eq. (17) by $P_\mu/(-M^2)$, we can express the leptonic decay constant f_P as,

$$f_P = f_P^{(0)} + f_P^{(1)} + f_P^{(2)} + f_P^{(3)}, \quad (19)$$

where $f_P^{(0)}$, $f_P^{(1)}$, $f_P^{(2)}$, $f_P^{(3)}$, are the contributions to f_P from the four Dirac covariants asso-

ciated with coefficients B_i ($i = 0, 1, 2, 3$), and are expressed as:

$$\begin{aligned}
f_P^{(0)} &= \sqrt{3}N_P B_0 \int d^3\hat{\mathbf{q}} D(\hat{q})\phi(\hat{q}) \int_{-\infty}^{\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[-2m_1 + 2\frac{m_1^3}{M^2} - 2m_2 - 2\frac{m_1^2 m_2}{M^2} - 2\frac{m_1 m_2^2}{M^2} \right. \\
&\quad \left. + 2\frac{m_2^3}{M^2} + 4(m_1 - m_2)\sigma \right], \\
f_P^{(1)} &= \sqrt{3}N_P B_1 \int d^3\hat{\mathbf{q}} D(\hat{q})\phi(\hat{q}) \int_{-\infty}^{\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[M - \frac{m_1^4}{M^3} + 4\frac{m_1 m_2}{M} + 2\frac{m_1^2 m_2^2}{M^3} - \frac{m_2^4}{M^3} - 4\frac{\hat{q}^2}{M} \right. \\
&\quad \left. + (m_2^2 - m_1^2)\sigma \frac{4}{M} - 4M\sigma^2 \right], \\
f_P^{(2)} &= \sqrt{3}N_P B_2 \int d^3\hat{\mathbf{q}} D(\hat{q})\phi(\hat{q}) \int_{-\infty}^{\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left[\frac{4}{M^3}(m_1^2 - m_2^2)\hat{q}^2 \right. \\
&\quad \left. + \left(M - \frac{m_1^4}{M^3} + 4\frac{m_1 m_2}{M} + 2\frac{m_1^2 m_2^2}{M^3} - \frac{m_2^4}{M^3} \right) \sigma + 4\frac{\hat{q}^2}{M}\sigma^2 + \frac{4}{M}(m_2^2 - m_1^2)\sigma^2 - 4M\sigma^3 \right], \\
f_P^{(3)} &= \sqrt{3}N_P B_3 \int d^3\hat{\mathbf{q}} D(\hat{q})\phi(\hat{q}) \int_{-\infty}^{\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \left(-8\frac{m_1 + m_2}{M^2}\hat{q}^2 \right).
\end{aligned} \tag{20}$$

In deriving the above expressions, we had made use of the scalar products of various momenta expressed in terms of integration variables \hat{q} and σ as,

$$\begin{aligned}
p_1 \cdot p_2 &= -M^2(\hat{m}_1 + \sigma)(\hat{m}_2 - \sigma) - \hat{q}^3, \\
p_1 \cdot P &= -M^2(\hat{m}_1 + \sigma), \\
p_2 \cdot P &= -M^2(\hat{m}_2 - \sigma), \\
P \cdot q &= -M^2\sigma, \\
p_1^2 &= -M^2(\hat{m}_1 + \sigma)^2 + \hat{q}^2, \\
p_2^2 &= -M^2(\hat{m}_2 - \sigma)^2 + \hat{q}^2, \\
p_1 \cdot q &= \frac{1}{2}\{2\hat{q}^2 - \sigma[m_1^2 - m_2^2 + M^2(1 + 2\sigma)]\}, \\
p_2 \cdot q &= \frac{1}{2}\{-2\hat{q}^2 + \sigma[m_1^2 - m_2^2 + M^2(-1 + 2\sigma)]\}.
\end{aligned} \tag{21}$$

We see that on the right hand side of the expression for f_P , each of the expressions multiplying the constant parameters B_0 and B_1 consist of two parts, of which only the second part explicitly involves the off-shell parameter σ . It is can be seen that the off-shell contribution which vanishes for $m_1 = m_2$ in case of using only the leading covariant γ_5 , would no longer vanish for $m_1 = m_2$ in the above calculation for f_P (when other covariants are

incorporated in $Hq\bar{q}$ vertex function besides the leading covariant γ_5) due to the terms like $4M$ and $4\hat{q}^2/M$ multiplying σ^2 in $f_P^{(2)}$ and $f_P^{(3)}$ respectively. This possibly implies that when other covariants besides γ_5 are incorporated into the vertex function, the off-shell part of f_P does not arise from unequal mass kinematics alone (which is in complete contrast to the earlier CIA calculation of f_P employing only γ_5). This may be a pointer to the fact that Dirac covariants other than γ_5 might also be important for the study of processes involving large q^2 (off-shell). Carrying out integration over $d\sigma$ by method of contour integration by noting the pole positions in the complex σ -plane:

$$\begin{aligned}\Delta_1 = 0 &\Rightarrow \sigma_1^\pm = \pm \frac{\omega_1}{M} - \hat{m}_1 \mp i\varepsilon, \quad \omega_1^2 = m_1^2 + \hat{q}^2, \\ \Delta_2 = 0 &\Rightarrow \sigma_2^\mp = \mp \frac{\omega_2}{M} + \hat{m}_2 \pm i\varepsilon, \quad \omega_2^2 = m_2^2 + \hat{q}^2,\end{aligned}\tag{22}$$

we can again express f_P as $f_P = f_P^{(0)} + f_P^{(1)} + f_P^{(2)} + f_P^{(3)}$, where now

$$\begin{aligned}
f_P^{(0)} &= \sqrt{3}N_P B_0 \int d^3\hat{\mathbf{q}} D(\hat{q}) \phi(\hat{q}) \left[\left(-2m_1 + 2\frac{m_1^3}{M^2} - 2m_2 - 2\frac{m_1^2 m_2}{M^2} - 2\frac{m_1 m_2^2}{M^2} \right. \right. \\
&\quad \left. \left. + 2\frac{m_2^3}{M^2} \right) \frac{1}{D(\hat{q})} + 4(m_1 - m_2)R_1 \right], \\
f_P^{(1)} &= \sqrt{3}N_P B_1 \int d^3\hat{\mathbf{q}} D(\hat{q}) \phi(\hat{q}) \left[\left(M - \frac{m_1^4}{M^3} + 4\frac{m_1 m_2}{M} + 2\frac{m_1^2 m_2^2}{M^3} - \frac{m_2^4}{M^3} \right) \frac{1}{D(\hat{q})} \right. \\
&\quad \left. - 4\frac{\hat{q}^2}{M} \frac{1}{D(\hat{q})} + \frac{4}{M}(m_2^2 - m_1^2)R_1 - 4MR_2 \right], \\
f_P^{(2)} &= \sqrt{3}N_P B_2 \int d^3\hat{\mathbf{q}} D(\hat{q}) \phi(\hat{q}) \left[\frac{4}{M^3}(m_1^2 - m_2^2)\hat{q}^2 \frac{1}{D(\hat{q})} \right. \\
&\quad \left. + \left(M - \frac{m_1^4}{M^3} + 4\frac{m_1 m_2}{M} + 2\frac{m_1^2 m_2^2}{M^3} - \frac{m_2^4}{M^3} \right) R_1 \right. \\
&\quad \left. + 4\frac{\hat{q}^2}{M} R_1 + \frac{4}{M}(m_2^2 - m_1^2)R_2 \right], \\
f_P^{(3)} &= \sqrt{3}N_P B_3 \int d^3\hat{\mathbf{q}} D(\hat{q}) \phi(\hat{q}) \left[-8\frac{1}{M^2}(m_1 + m_2)\hat{q}^2 \frac{1}{D(\hat{q})} \right],
\end{aligned} \tag{23}$$

and $D(\hat{q})$ is given in Eq. (6), and the results of σ -integration in the complex σ -plane, on whether the contour is closed from above or below the real σ -axis is:

$$\begin{aligned}
R_1 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \sigma = \frac{M^2(-\omega_1 + \omega_2) + (m_1^2 - m_2^2)(\omega_1 + \omega_2)}{4M^2\omega_1\omega_2[M^2 - (\omega_1 + \omega_2)^2]}, \\
R_2 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2} \sigma^2 \\
&= \frac{(-M^4 - m_{12}^2 \delta m^2 + 4M^2\omega_1\omega_2)(\omega_1 + \omega_2) + 2M^2 m_{12} \delta m(\omega_2 - \omega_1)}{8M^4\omega_1\omega_2[M^2 - (\omega_1 + \omega_2)^2]}.
\end{aligned} \tag{24}$$

To calculate BS normalizer N_P for a pseudoscalar meson in the expression for f_P in Eq.

(23), we use the current conservation condition [2],

$$2iP_\mu = (2\pi)^4 \int d^4q \text{Tr} \left[\bar{\Psi}(P, q) \left(\frac{\partial}{\partial P_\mu} S_F^{-1}(p_1) \right) \Psi(P, q) S_F^{-1}(-p_2) \right] + (1 \Leftrightarrow 2). \quad (25)$$

Putting BS wave function $\Psi(P, q)$ from Eq. (18) in the above equation, carrying out derivatives of inverse of propagators of constituent quarks with respect to total momentum of hadron P_μ , evaluating trace over the gamma matrices, following usual steps and multiplying both sides of equation by $P_\mu/(-M^2)$ to extract out the normalizer N_P from the above expression, we then express the above expression in terms of integration variables \hat{q} and σ . Noting that the four dimensional volume element $d^4q = d^3\hat{q}M d\sigma$, we then perform pole integration over $d\sigma$ in complex σ -plane, making use of the pole positions in Eq. (22). The calculation of normalizer is extremely complex due to unequal mass kinematics. We thus give here a general expression for the normalizer integral of the form,

$$N_P^{-1} = -(2\pi)^2 i \int d^3\hat{q} D^2(\hat{q}) \phi^2(\hat{q}) [g_1(B, \hat{q}) I_1 + g_2(B, \hat{q}) I_2 + g_3(B, \hat{q}) I_3 + g_4(B, \hat{q}) I_4], \quad (26)$$

where $B \equiv (B_0, B_1, B_2, B_3)$ and g_1, \dots, g_4 are extremely complicated functions of B and \hat{q} and are extremely lengthy expressions, and hence we do not present their actual forms here, whereas I_1, \dots, I_4 are analytic results of pole integration over the off-shell variable σ in the complex σ -plane and are expressed as:

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} = 2\pi i \left[\frac{2\omega_1^3 - M^2\omega_2 + 5\omega_1^2\omega_2 + 4\omega_1\omega_2^2 + \omega_2^3}{4\omega_1^3\omega_2(M^2 - (\omega_1 + \omega_2)^2)^2} \right], \\ I_2 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} \sigma \\ &= 2\pi i \frac{-M^4\omega_2 + (m_1^2 - m_2^2)(\omega_1 + \omega_2)^2(2\omega_1 + \omega_2)M^2[6\omega_1^3 + 9\omega_1^2\omega_2 + 4\omega_1\omega_2^2 + \omega_2(-m_1^2 + m_2^2 + \omega_2^2)]}{8M^2\omega_1^3\omega_2[M^2 - (\omega_1 + \omega_2)^2]^2}, \end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} \sigma^2 \\
&= 2\pi i \frac{1}{16M^4 \omega_1^3 \omega_2 (-M^2 + (\omega_1 + \omega_2)^2)^2} \{ -M^6 \omega_2 + (m_1^2 - m_2^2)^2 (\omega_1 + \omega_2)^2 (2\omega_1 + \omega_2) \\
&\quad + M^4 [2\omega_1^3 - 2m_1^2 \omega_2 + 2m_2^2 \omega_2 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3] \\
&\quad - M^2 [m_1^4 \omega_2 + m_2^4 \omega_2 + 4\omega_1^2 \omega_2 (\omega_1 + \omega_2)^2 + 2m_2^2 (-2\omega_1^3 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3) \\
&\quad - 2m_1^2 (-2\omega_1^3 + m_2^2 \omega_2 + \omega_1^2 \omega_2 + 4\omega_1 \omega_2^2 + \omega_2^3)] \} \\
I_4 &= \int_{-\infty}^{+\infty} \frac{M d\sigma}{\Delta_1^2 \Delta_2} \sigma^3 \\
&= 2\pi i \left\{ \frac{(M^2 - m_1^2 + m_2^2 + 2M\omega_2)^3}{8M^6 \omega_2 (M^2 - \omega_1^2 + 2M\omega_2 + \omega_2^2)^2} \right. \\
&\quad + \frac{(M^2 + m_1^2 - m_2^2 - 2M\omega_1)^2 [M^4 + M^2(m_1^2 - m_2^2 - \omega_1^2 - \omega_2^2) + (m_1^2 - m_2^2)(3\omega_1^2 - \omega_2^2)]}{16M^6 \omega_1^3 (M^2 - 2M\omega_1 + \omega_1^2 - \omega_2^2)^2} \Big\} \\
&\quad + \frac{(M^2 + m_1^2 - m_2^2 - 2M\omega_1)^2 [-4M\omega_1(m_1^2 - m_2^2 + \omega_2^2)]}{16M^6 \omega_1^3 (M^2 - 2M\omega_1 + \omega_1^2 - \omega_2^2)^2} \Big\}. \tag{27}
\end{aligned}$$

After this, numerical integration over the 3-D variable $d^3\hat{\mathbf{q}}$ in Eq. (26) is performed to evaluate N_P .

We have thus evaluated the expressions for f_P and N_P in framework of BSE under CIA, with Dirac structures of eq. (10). introduced in the $Hq\bar{q}$ vertex function besides γ_5 according to our power counting rule. We see that so far the results are independent of any model for $\phi(\hat{q})$. However, for calculating the numerical values of these decay constants one needs to know the constant coefficients B_0, B_1, B_2, B_3 which are associated with the above Dirac structures. Because of the normalization condition, we take $B_0 = 1$, and then there are 3 parameters $B_1/B_0, B_2/B_0, B_3/B_0$, which will still be denoted as B_1, B_2, B_3 for simplicity. To

see the contribution of various Dirac covariants on the calculation of meson decay constants, we first discuss the numerical procedure adopted to fit these coefficients

B. Numerical Calculation

Eq. (23) which expresses decay constants f_P of pseudo-scalar mesons in terms of the parameters B_0, B_1, B_2, B_3 is a highly non linear function of the B_i 's. This obviously implies that numerical methods must be applied to solve the problem.

We used a simple Mathematica procedure for calculating the numerical integrals and searching for accurate values of the B_i ($i = 0, \dots, 3$). We defined the following auxiliary function $W(B)$ which is positive definite as,

$$W(B) = \sum_P [f_P(B) - f_P(exp.)]^2, \quad (28)$$

where $B \equiv (B_0, B_1, B_2, B_3)$, and summation in the above equation runs over five pseudoscalar mesons π, K, D, D_S and B mesons studied in this work, and $f_P(exp.)$ are the central values of experimental data on decay constants [13, 14] (indicated in Table II).

From the numerical point of view the problem reduces to finding values of B_i 's such that $W(B)$ has a minimum. We used Mathematica package which has some useful functions for minimizing. Those functions start from a point and search for a minimum near to that initial point. We constrained all the B_i 's to lie within the interval $[0,1]$. We generated in a random way values of the B_i in this interval. Starting from those values, the Mathematica minimization function finds a minimum. Then it is checked if this minimum is "sufficiently near to zero". This check is done by evaluating the percent average of the absolute values of the differences between the predicted f_P values from the experimental value $f_P(exp.)$. Using this method we found that the values of coefficients B_0, \dots, B_3 (with average error with respect to the experimental data less than 3.5%) respectively are: $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ to give the decay constant values, $f_\pi = 0.130$ GeV, $f_K = 0.164$ GeV, $f_D = 0.194$ GeV, $f_{D_s} = 0.296$ GeV. and $f_B = 0.228$ GeV which are within the error bars of experimental data [13, 14] depicted in Table II for these five pseudoscalar mesons. These values of f_P along with the contributions from various covariants and comparison with various models and experimental results are listed in Tables I and II.

There is one important point which needs to be clarified: The experimental data have different error bar, e.g., the data of π has a very high precision to the order of 0.1%, while for the case of B , the relative error is more than 16%. So in the fitting, we should take into account the difference, e.g., assign different weight for these data. However, we only give our formulation at NLO. From the above discussions, it is straightforward to recognize, the smaller the meson mass, the larger the contributions of higher orders. For the case of pion, we even can perspect that higher order contributions (coming from higher order terms of Taylor series of B'_i s as powers of $\frac{q.P}{M^2}$) could be also very important. So, it is not reasonable to expect the NLO formulae can fit the data of pion to the precision of order of 0.1%. This is the reason why we fit the central value of data equally, as described above.

IV. RADIATIVE DECAY CONSTANTS OF NEUTRAL P-MESONS

In this section we calculate the radiative decays of a neutral pseudoscalar meson such as π^0 or η_c proceeding through the process $P \longrightarrow \gamma\gamma$ which proceed through the famous quark-triangle diagrams in the above framework using both the leading order and the next-to-leading order covariants in the Hadron-quark vertex function, taking the values of parameters $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ fixed above in the calculation of f_P values of π , K , D , D_S and B mesons. The invariant amplitude for the decay of a neutral P-meson into two photons can be expressed as summation over the two triangle diagrams corresponding to the Direct and Exchange processes as:

$$A(P \rightarrow 2\gamma) = \frac{e^2}{\sqrt{6}} \int d^4q Tr[\bar{\Psi}(P, q) i\gamma \cdot \epsilon_1 S_F(q-Q) i\gamma \cdot \epsilon_2] + \frac{e^2}{\sqrt{6}} \int d^4q Tr[\bar{\Psi}(P, q) i\gamma \cdot \epsilon_2 S_F(q+Q) i\gamma \cdot \epsilon_1] \quad (29)$$

where $\Psi(P, q)$ is the BS wave function of a neutral P-meson given explicitly in Eq.(20) and Eq.(12)- (13), $S_F(q \pm Q)$ are the propagators of the third quark in the Direct and Exchange diagrams respectively, where $Q = k_1 - k_2$ is the the difference in momenta of the two emitted photons with momenta k_1 and k_2 respectively, while $\epsilon_{1,2}$ are the polarization vectors of the two emitted photons in the above diagrams which differ from each other in the interchange $1 \Leftrightarrow 2$. Evaluating traces over the gamma- matrices, combining various terms and then performing pole-integrations in the complex σ -plane, we

can express amplitude for the above process as:

$$A(P \rightarrow 2\gamma) = [F_P] \epsilon_{\mu\nu\rho\sigma} P_\mu \epsilon_{2\nu} Q_\rho \epsilon_{1\sigma}, \quad (30)$$

where $P = p_1 + p_2$ is the total hadron momentum, where $p_{1,2}$ are the momenta of the quarks constituting the hadron, and the radiative decay constant, F_P is given as (the B_2 term vanishes because of wrong charge parity),

$$F_P = \frac{e^2 N_P}{\sqrt{6}} \int d^3\hat{q} D(\hat{q}) \phi(\hat{q}) \left[B_0[8mS_1] + B_1\left[\frac{-16m^2}{M}S_1 + \frac{4}{M}S_2 + \frac{4}{M}S_3\right] + B_3\left[\frac{8m}{M^2}(S_2 + S_3 + S_4 - S_5)\right] \right], \quad (31)$$

where $S_{1,2,3,4,5}$ are the analytical results of integrals

over the off-shell parameter σ :

$$S_1 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_2 \Delta_3} = \frac{12}{M^4 \omega - 20M^2 \omega^3 + 64\omega^5}; \quad (32)$$

$$S_2 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_2 \Delta_3} = \frac{4}{-M^2 \omega + 16\omega^3}; \quad (33)$$

$$S_3 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_3} = \frac{4}{-M^2 \omega + 16\omega^3};$$

$$S_4 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_1 \Delta_3} \sigma = \frac{-1}{-M^2 \omega + 16\omega^3} \quad (34)$$

$$S_5 = \int_{-\infty}^{+\infty} \frac{M d\sigma}{2\pi i \Delta_2 \Delta_3} \sigma = \frac{1}{-M^2 \omega + 16\omega^3} \quad (35)$$

evaluated by the method of contour integrations by noting the various pole positions in the complex σ -plane :

$$\begin{aligned}
\Delta_1 = 0 &\Rightarrow \sigma_1^\pm = \pm \frac{\omega}{M} - \frac{1}{2} \mp i\varepsilon; \\
\Delta_2 = 0 &\Rightarrow \sigma_2^\pm = \pm \frac{\omega}{M} + \frac{1}{2} \mp i\varepsilon; \\
\Delta_3 = 0 &\Rightarrow \sigma_3^\pm = \pm \frac{\omega}{M} \mp i\varepsilon; \omega^2 = m^2 + \hat{q}^2
\end{aligned} \tag{36}$$

corresponding to inverse propagators of the three quarks (of which $\Delta_{1,2}$ correspond to the two constituent quarks in the meson) in the quark-triangle diagrams, expressed in terms of the off-shell parameter σ as:

$$\begin{aligned}
\Delta_1 &= \omega^2 - M^2\left(\frac{1}{2} + \sigma\right); \\
\Delta_2 &= \omega^2 - M^2\left(\frac{1}{2} - \sigma\right); \\
\Delta_3 &= \omega^2 - M^2\sigma^2
\end{aligned} \tag{37}$$

From Eq.(33), it can be noticed that the contribution to radiative decay constant F_P from one of the next-to-leading order covariants associated with the parameter B_2 completely vanishes after trace evaluation. Numerical evaluation of F_P for π^0 and η_c using the same set of parameters, B_i / B_0 fixed from the calculation of leptonic decay constant f_P values of π, K, D, D_S and B mesons above gives $F_\pi = .031 GeV^{-1}$, $F_{\eta_c} = .006 GeV^{-1}$. These are very close to the experimental numbers $F_\pi(Exp.) = .025 GeV^{-1}$, and $F_{\eta_c}(Exp.) = .0074 GeV^{-1}$ which are arrived at through the expression, $\Gamma(P \rightarrow 2\gamma) = \frac{F_P^2 M^3}{64\pi}$ connecting the decay width Γ with radiative decay constants, F_P , using the central values of experimental data on decay widths for π and η_c mesons as $\Gamma(\pi^0 \rightarrow 2\gamma) = 8.5 eV$ and $\Gamma(\eta_c \rightarrow 2\gamma) = 7.4 KeV$ [17, 18] respectively.

V. DISCUSSION

In this paper we have calculated the decay constants f_P of pseudoscalar mesons π , K , D , D_S and B and radiative decay constants F_P for neutral pseudoscalar mesons π^0 and η_c proceeding through the process $P \rightarrow 2\gamma$ in BSE under CIA. The Hadron-quark vertex function incorporates various Dirac covariants order-by-order in powers of inverse of meson mass within its structure in accordance with a power counting rule from their complete set. This power counting rule suggests that the maximum contribution to any meson observable should come from Dirac structures associated with Leading order terms alone, followed by Dirac structures associated with Next-to-Leading Order terms in the vertex function. Incorporation of all these covariants is found to bring calculated f_P values much closer to results of experimental data [13, 14] and some recent calculations [5, 6, 15, 16] for π , K , D , D_S and B mesons. The f_P are within the error bars of experimental data for each one of these five mesons by fitting three parameters. The calculation of radiative decay constants of π^0 and η_c is again close to the experimental data[17, 18].

The results for π , K , D , D_S and B mesons with parameter set: $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ (giving f_P values with average error with respect to experimental data less than 3.5%) are presented in Table I. In Fig. 1 we are plotting functions $I_P^i(\hat{q})$ ($i = 0, \dots, 3$) vs \hat{q} , where $I_P^i(\hat{q})$ is the integrand of $f_P^{(i)}$ in equations (23). The plots of variations of $I_P^0(\hat{q}), \dots, I_P^3(\hat{q})$ with \hat{q} for π , K , D , D_S and B mesons, along with the results in Table I, show that the contribution to f_P from NLO covariants is much smaller than the contribution from LO covariants for K , D , D_S and B mesons. Comparison with experimental data and other models is shown in Table II. It is seen from Table I that as far as the various contributions to decay constants f_P are concerned, for K mesons, the LO terms contribute 60%, while NLO terms 40%. However for heavy-light meson D , the LO contribution increases to 90%, while NLO contribution is 10%. For D_S meson, LO contribution is 91%, while NLO contribution is 9%. But for B meson, the LO contribution is 96%, while NLO contribution reduces to just 4%. This is in conformity with the power counting rule according to which the leading order covariants, γ_5 and $i\gamma_5(\gamma \cdot P)(1/M)$ (associated with coefficients B_0 and B_1) should contribute maximum to decay constants followed by the next-to-leading order covariants, $-i\gamma_5(\gamma \cdot q)(1/M)$ and $-\gamma_5[(\gamma \cdot P)(\gamma \cdot q) - (\gamma \cdot q)(\gamma \cdot P)](1/M^2)$ (associated with coefficients B_2 and B_3) in the BS wave function, Eq. (9)-(10).

However the situation is different for the lightest meson π which enjoys a unique status due to the fact that mass of a pion, M is unusually small ($\ll \Lambda_{QCD}$), and the large difference between the sum of two constituent quark masses and the pion mass shows that the quarks are far off shell and the internal momentum q should be the same order as the pion mass and the approximation $q \ll P \sim M$ breaks down for pion. Hence the contribution of NLO covariants in pion case is even larger than the contribution of LO covariants. Thus, the NLO covariants in pion should play a more dominant role in contrast to heavier mesons K , D , D_S and B . However the sum of LO and NLO contributions adds up to the experimental value for pion $f_P (=0.130 \text{ GeV})$. Further investigations on higher order terms can show even more details of the pion structure.

To check the validity of our calculation, we then do numerical evaluation of radiative decay constants F_P for π^0 and η_c using the same set of parameters, B_i/B_0 fixed above from the calculation of leptonic decay constant f_P values of π, K, D, D_S and B mesons. This gives $F_\pi = .031 \text{ GeV}^{-1}$, $F_{\eta_c} = .006 \text{ GeV}^{-1}$. These are very close to the experimental numbers $F_\pi(Exp.) = .025 \text{ GeV}^{-1}$, and $F_{\eta_c}(Exp.) = .0074 \text{ GeV}^{-1}$ which are arrived at through the expression, $\Gamma(P \rightarrow 2\gamma) = \frac{F_P^2 M^3}{64\pi}$ connecting the decay width Γ with radiative decay constants, F_P , using the central values of experimental data on decay widths for π and η_c mesons as $\Gamma(\pi^0 \rightarrow 2\gamma) = 8.5 \text{ eV}$ and $\Gamma(\eta_c \rightarrow 2\gamma) = 7.4$ [17, 18] respectively.

The numerical results for leptonic decay constants, f_P and radiative decay constants, F_P obtained in our framework upto the next to leading order covariants demonstrates the validity of our power counting rule, which also provides a practical means of incorporating various Dirac covariants in the BS wave function of a hadron. By this rule, we also get to understand the relative importance of various covariants to calculate various meson observables. This would in turn help in obtaining a better understanding of the hadron structure. Here would like mention the robustness of our framework: On one hand, at lower order(s), with limited number of parameters, we can globally reproduce almost all the decay constants of certain kinds of meson. On the other hand, by introducing higher order corrections, we can accommodate enough parameters to fit the data as precise as possible, so than to get a good parameterization of the structure of certain special hadron for further investigations.

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	f_P^0	f_P^1	f_P^2	f_P^3	$ f_P^{LO} $	$ f_P^{NLO} $	$f_P^{LO}(\%)$	$f_P^{NLO}(\%)$	$f_P = f_P^{LO} + f_P^{NLO}$
π	0.110	-0.154	0.000	0.175	0.044	0.175	25%	75%	0.130
K	0.202	-0.104	0.025	0.039	0.098	0.064	60%	40%	0.164
D	0.271	-0.097	0.010	0.009	0.174	0.019	90%	10%	0.194
D_S	0.426	-0.156	0.013	0.013	0.270	0.026	91%	9%	0.296
B	0.345	-0.125	0.005	0.003	0.220	0.008	96%	4%	0.228

TABLE I: Decay constant f_P values (in GeV) for π , K , D , D_S and B mesons in BSE with the individual contributions f_P^0 , f_P^1 , f_P^2 , f_P^3 from various Dirac covariants along with the contributions from LO and NLO covariants and also their % contributions for parameter set: $B_0 = 1$, $B_1/B_0 = 0.3727$, $B_2/B_0 = 0.2234$, $B_3/B_0 = 0.0821$ (with average error with respect to the experimental data less than 3.5%)

	f_π	f_K	f_D	f_{D_S}	f_B
BSE (3.5% average error) present paper	0.130	0.164	0.194	0.296	0.228
BSE [5]				0.248	
SDE [6]		0.164			
Lattice [15]			0.208 ± 0.004	0.241 ± 0.003	
QCD-SR [16]			0.20 ± 0.02	0.23 ± 0.02	
Exp. Results [13]	0.1300 ± 0.0001	0.159 ± 0.001	0.22 ± 0.02	0.29 ± 0.03	
Babar+Belle Collaboration [14]					0.24 ± 0.04

TABLE II: Comparison of results of f_P (in GeV) for π , K , D , D_S and B in BSE with the parameter set $B_0 = 0.7045$, $B_1 = 0.2626$, $B_2 = 0.1574$, $B_3 = 0.0579$ (with average error 3.5%) with those of other models and experimental data.







